

# Quasi-power laws in multiparticle production processes<sup>☆</sup>

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## Abstract

We review the ubiquitous presence in multiparticle production processes of quasi-power law distributions (i.e., distributions following pure power laws for large values of the argument but remaining finite, usually exponential, for small values). Special emphasis is placed on the conjecture that this reflects the presence in the produced hadronic systems of some intrinsic fluctuations. If described by parameter  $q$  they form, together with the scale parameter  $T$  ("temperature"), basis of Tsallis distribution,  $f(X) \sim [1 - (1 - q)X/T]^{1/(1-q)}$ , frequently used to describe the relevant distributions (the  $X$  being usually a transverse momentum). We discuss the origin of such quasi-power law behavior based on our experience with the description of multiparticle production processes. In particular, we discuss Tsallis distribution with complex nonextensivity parameter  $q$  and argue that it is needed to describe log-oscillations as apparently observed in recent data on large momentum distributions in very high energy p-p collisions.

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## 1. Introduction

Multiparticle production processes, which will serve us as the stage to present and discuss the ubiquitous presence of the quasi-power law distributions, cover at present a  $\sim 14$  orders of magnitude span in the observed cross sections (when one observes transverse momentum,  $p_T$  distributions [1, 2, 3]). It came as a surprise that this type of data can be fitted by a one simple *quasi-power like* formula [4, 5]:

$$H(X) = C \cdot \left(1 + \frac{X}{nX_0}\right)^{-n} \longrightarrow \begin{cases} \exp\left(-\frac{X}{X_0}\right) & \text{for } X \rightarrow 0, \\ X^{-n} & \text{for } X \rightarrow \infty, \end{cases} \quad (1)$$

which smoothly combines pure power-like behavior in one part of phase space with an exponential in another part. Before proceeding further, a few words of explanation are in order. Fig. 1 displays our playground, i.e., the phase space for particle produced in high energy collision of, say protons,  $p + p \rightarrow N$  particles (mostly  $\pi$  mesons). The initial energy is  $E_{in}$  and momenta of colliding particles are  $\vec{p}_A$  and  $\vec{p}_B$ . One usually works in center-of-mass system in which, for  $pp$  collisions,  $\vec{p}_A = -\vec{p}_B$  and  $|\vec{p}_{A,B}| = P$ . The momenta of produced secondaries are decomposed in

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the longitudinal and perpendicular components (with regards to the collision axis),  $\vec{p} = [p_L, \vec{p}_T]$ ; their energies are  $E = (\mu^2 + p^2)^{1/2}$ , where  $\mu$  is mass of the produced secondary and  $p = |\vec{p}|$ . Composition  $(\mu^2 + |\vec{p}_T|^2)^{1/2} = \mu_T$  is called transverse mass. It is also customary to use variable  $y = \ln[(E + p_L)/(E - p_L)]$  (rapidity) in which  $p_L = \mu_T \sinh y$  and  $E = \mu_T \cosh y$ . In what follows, we shall be interested in the so called central rapidity region (with  $y \simeq 0$ , i.e., with  $p_L \simeq 0$ ), and in distributions of  $p_T$  or  $\mu_T$  only (i.e., in Eq. (1)  $X = p_T$  or  $X = \mu_T$ ).

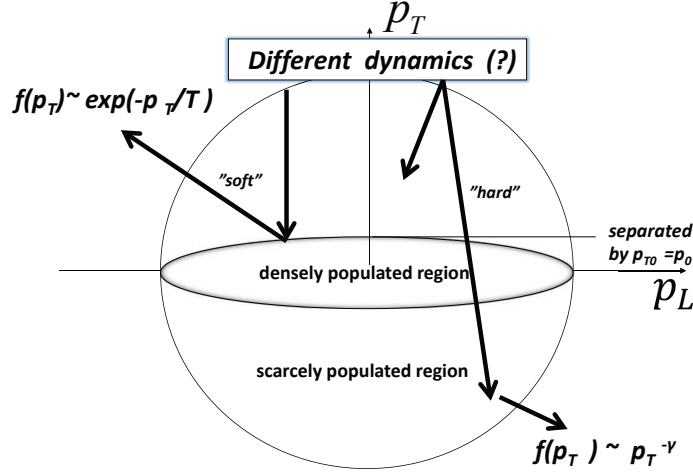


Figure 1. Schematic view of details of the phase space in multiparticle production. Cf. text for details..

Different parts of phase space depicted schematically in Fig. 1. They are dominated by different collision dynamics. It is customary to separate them artificially by some momentum scale, transverse momentum parameter  $p_0$ , dividing transverse phase space into a predominantly "hard" and predominantly "soft" part. They distinguished themselves by the type of observed spectra of secondaries produced. In the "hard" (scarcely populated) region, with  $p_T > p_0$ , they are regarded as essentially power-like,  $F(p_T) \sim p_T^{-n}$ , and are usually associated with the hard scattering processes between partons (constituents composing nucleons, quarks and gluons) [6]. In the "soft" (densely populated) region, with  $p_T < p_0$ , the dominant distribution is exponential one,  $F(p_T) \sim \exp(-p_T/T)$ . It is usually associated with the thermodynamical description of the hadronizing system with  $T$  playing the role of "temperature", with the fragmentation of a flux tube with a transverse dimension, or with the production of particles by the Schwinger mechanism [6]. However, both formulas can be unified in a single quasi-power like formula, Eq. (1), interpolating smoothly between both regions. It becomes power-like for high  $p_T$  and exponential-like for low  $p_T$ , as required. One way of introducing it is to start from the very large values of  $p_T$  where we have a pure, scale free, power law. Decreasing now  $p_T$  towards the demarcation value  $p_0$  and below it, the natural thing to avoid problems with unphysical singularity for  $p_T \rightarrow 0$  is to add to  $p_T$  a constant term, and choose it equal to  $p_0$ . In this way one gets Eq. (1), which for small values of  $X = p_T$  (and for  $X_0 = p_0$ ) becomes exponential (Boltzmann-Gibbs - BG) distribution with temperature  $T = p_0^{-1}$ .

This formula coincides with the so called Tsallis nonextensive distribution [8] for  $n = 1/(q - 1)$ ,

$$h_q(X) = C_q \cdot \left[ 1 - (1 - q) \frac{X}{X_0} \right]^{\frac{1}{1-q}} \stackrel{\text{def}}{=} C_q \cdot \exp_q \left( -\frac{X}{X_0} \right) \stackrel{q \rightarrow 1}{\Rightarrow} C_1 \cdot \exp \left( -\frac{X}{X_0} \right). \quad (2)$$

It has been widely used in many other branches of physics [9]. For our purposes, both formulas are equivalent with  $n = 1/(q - 1)$  and  $X_0 = nT$ , and we shall use them interchangeably. Because Eq. (2) describes nonextensive systems

<sup>1</sup>Formula (1) is known as the *QCD-based Hagedorn formula* [5]. It was used for the first time in the analysis of UA1 experimental data [7] and it became one of the standard phenomenological formulas for  $p_T$  data analysis.

in statistical mechanics, the parameter  $q$  is usually called the *nonextensivity parameter*. Eq. (2) becomes the usual Boltzmann-Gibbs exponential distribution for  $q \rightarrow 1$ , with  $T$  becoming the temperature. Both Eqs. (1) and (2) have been widely used in data analysis (cite, for example, [1, 2, 3, 7, 10, 11]) and in the phenomenological analysis of processes of multiparticle production (cf., for example [12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22]).

In the next Section 2 we present a review of a number of examples of how such distributions occur (based on our experience with applications of Tsallis statistics to multiparticle production processes, but they have general applicability). In Section 3.1 we present some specific generalization of quasi-power distributions which allows to account for a log-periodic oscillations in variable considered. This phenomenon was so far known and observed in all other situations resulting in power-like distributions. However, recent multiparticle production data [1, 2, 3] seem to indicate that, apparently, this phenomenon also starts to be visible there and deserves attention. Section 4 contains a summary.

## 2. How to get a Tsallis distribution - some examples

In what follows we shall concentrate on Tsallis distributions, Eq. (2), obtained from approaches not based on in nonextensive thermodynamics<sup>2</sup>. We start from examples of constrained systems which lead to Tsallis distributions with  $q < 1$ . To get also  $q > 1$  one has to allow for some intrinsic fluctuations (or relaxing constraints). This will be shown next.

### 2.1. All variables fixed

In statistical physics, the simplest situation considered is when a system is characterized by variables  $U$  - energy,  $T$  - temperature and  $N$  - multiplicity. Usually one or two of them are fixed and the rest fluctuates (either according to gamma distribution, in case of  $U$  or  $T$ , or according to Poisson distribution in case of  $N$ , which are integers) [24]. Only in the thermodynamic limit (i.e., for  $N \rightarrow \infty$ ) fluctuations take the form of Gaussian distributions usually discussed in textbooks. In [24] we also discussed in detail situations when all three variables fluctuate inducing some correlations in the system.

However, if all variables are fixed we have distributions of the type of

$$f(E) = \left(1 - \frac{E}{U}\right)^{N-2} \quad \text{with} \quad q = \frac{N-3}{N-2} < 1, \quad (3)$$

i.e., Tsallis distributions with  $q < 1$ . Interestingly enough, such a distribution emerges also directly from the calculus of probability for a situation known as *induced partition* [25]. In short:  $N - 1$  randomly chosen independent points  $\{U_1, \dots, U_{N-1}\}$  break segment  $(0, U)$  into  $N$  parts, the length of which is distributed according to Eq. (3). The length of the  $k^{\text{th}}$  part corresponds to the value of energy  $E_k = U_{k+1} - U_k$  (for ordered  $U_k$ ). One could think of some analogy in physics to the case of random breaks of string in  $N - 1$  points in the energy space. Notice that the induced partition differs from *successive sampling* from the uniform distribution,  $E_k \in [0, U - E_1 - E_2 - \dots - E_{k-1}]$ , which results in  $f(E) = 1/E$  [26].

### 2.2. Conditional probability

To the category of constrained systems also belongs an example of conditional probability. Consider a system of  $n$  independent points with energies  $\{E_{i=1, \dots, N}\}$ , each energy distributed according to a Boltzmann distribution  $g_i(E_i)$  (i.e., their sum,  $U = \sum_{i=1}^N E_i$ , is then distributed according to a gamma distribution  $g_N(U)$ ):

$$g_i(E_i) = \frac{1}{\lambda} \exp\left(-\frac{E_i}{\lambda}\right) \quad \text{and} \quad g_N(U) = \frac{1}{\lambda(N-1)} \left(\frac{U}{\lambda}\right)^{N-1} \exp\left(-\frac{U}{\lambda}\right). \quad (4)$$

If the available energy is limited,  $U = N\alpha = \text{const}$ , the resulting *conditional probability*

$$f(E_i|U = N\alpha) = \frac{g_i(E_i) g_{N-1}(N\alpha - E_i)}{g_N(N\alpha)} = \frac{(N-1)}{N\alpha} \left(1 - \frac{1}{N} \frac{E_i}{\alpha}\right)^{N-2} = \frac{2-q}{\lambda} \left[1 - (1-q) \frac{E_i}{\lambda}\right]^{\frac{1}{1-q}}, \quad (5)$$

<sup>2</sup>These are discussed in [19, 20]; their theoretical justification is presented in [23].

becomes a Tsallis distribution with

$$q = \frac{N-3}{N-2} < 1 \quad \text{and} \quad \lambda = \frac{\alpha N}{N-1}. \quad (6)$$

### 2.3. Statistical physics considerations

Both above results arise more formally from *statistical physics considerations* of isolated systems with energy  $U = \text{const}$  and with  $\nu$  degrees of freedom ( $\nu$  particles). Choose a single degree of freedom with energy  $E \ll U$  (i.e., the remaining, or reservoir, energy is  $E_r = U - E$ ). If this degree of freedom is in a single, well defined, state then the number of states of the whole system is  $\Omega(U - E)$  and probability that the energy of the chosen degree of freedom is  $E$  is  $P(E) \propto \Omega(U - E)$ . Expanding (slowly varying)

$$\ln \Omega(U - E) = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\partial^{(k)} \ln \Omega}{\partial E_r^{(k)}}, \quad \text{with} \quad \beta = \frac{1}{k_B T} \stackrel{\text{def}}{=} \frac{\partial \ln \Omega(E_r)}{\partial E_r}, \quad (7)$$

around  $U$ , and keeping only the two first terms one gets

$$\ln P(E) \propto \ln \Omega(E) \propto -\beta E, \quad \text{or} \quad P(E) \propto \exp(-\beta E), \quad (8)$$

i.e., a Boltzmann distribution (for which  $q = 1$ ). On the other hand, one usually expects

$$\Omega(E_r) \propto \left(\frac{E_r}{\nu}\right)^{\alpha_1 \nu - \alpha_2} \quad (9)$$

(where  $\alpha_{1,2}$  are of the order of unity; we put  $\alpha_1 = 1$  and, to account for diminishing the number of states in the reservoir by one state,  $\alpha_2 = 2$ ) [27]. One can than write that

$$\frac{\partial^k \beta}{\partial E_r^k} \propto (-1)^k k! \frac{\nu - 2}{E_r^{k+1}} = (-1)^k k! \frac{\beta^{k-1}}{(\nu - 2)^k}. \quad (10)$$

Because

$$\ln(1 + x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{(k+1)},$$

the full series for probability of choosing energy  $E$  can be written as:

$$P(E) \propto \frac{\Omega(U - E)}{\Omega(U)} = \exp \left[ \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} \frac{1}{(\nu - 2)^k} (-\beta E)^{k+1} \right] = C \left( 1 - \frac{1}{\nu - 2} \beta E \right)^{(\nu-2)} = \beta(2 - q) [1 - (1 - q)\beta E]^{\frac{1}{1-q}}. \quad (11)$$

For  $q = 1 - 1/(\nu - 2) \leq 1$  this result coincides with the previous results from the induced partition and conditional probability.

### 2.4. Systems with fluctuating multiplicity $N$

So far, we were getting Tsallis distributions with  $q < 1$ . To obtain  $q > 1$  one has to relax the restrictions imposed on the system considered; for example, by allowing for fluctuations of one of the variables  $U$ ,  $N$  or  $T$ . Fluctuations of  $T$ , known as *superstatistics*, were considered in [28, 29, 30]. Here we shall consider multiplicity  $N$  fluctuating according to some distribution  $P(N)$ . In this case, the resulting distribution is

$$f(E) = \sum f_N(E) P(N), \quad (12)$$

where

$$f_N(E) = \left(1 - \frac{E}{U}\right)^N \quad \text{and} \quad U = \sum E = \text{const}, \quad (13)$$

is a distribution for fixed  $N$  (to simplify notation we changed  $N - 2$  in Eq. (3) to  $N$ ). The most characteristic for our purposes distributions  $P(N)$  are the, respectively, *Binomial Distribution*,  $P_{BD}$ , *Poissonian Distribution*,  $P_{PD}$  and *Negative Binomial Distributions*,  $P_{NBD}$  (cf. [31]):

$$P_{BD}(N) = \frac{M!}{N!(M-N)!} \left( \frac{\langle N \rangle}{M} \right)^N \left( 1 - \frac{\langle N \rangle}{M} \right)^{M-N}; \quad (14)$$

$$P_{PD}(N) = \frac{\langle N \rangle^N}{N!} e^{-\langle N \rangle}; \quad (15)$$

$$P_{NBD}(N) = \frac{\Gamma(N+k)}{\Gamma(N+1)\Gamma(k)} \left( \frac{\langle N \rangle}{k} \right)^N \left( 1 + \frac{\langle N \rangle}{k} \right)^{-k-N}. \quad (16)$$

They lead, respectively, to Tsallis distributions with  $q$  ranging from  $q < 1$  for the Binomial Distribution,  $P_{BD}$ , Eq.(17), via  $q = 1$  Boltzmann distribution for Poissonian Distribution,  $P_{PD}$ , Eq.(18), to  $q > 1$  for the Negative Binomial Distribution,  $P_{NBD}$ , Eq.(19) (in all cases  $\beta = \langle N \rangle / U$ ):

$$f_{BD}(E) = \left( 1 - \frac{\beta E}{M} \right)^M, \quad q = 1 - \frac{1}{M} < 1; \quad (17)$$

$$f_{PD}(E) = \exp(-\beta E), \quad q = 1; \quad (18)$$

$$f_{NBD}(E) = \left( 1 + \frac{\beta E}{k} \right)^{-k}, \quad q = 1 + \frac{1}{k} > 1. \quad (19)$$

In all three cases the physical meaning of the parameter  $q$  is the same: it measures the strength of multiplicity fluctuations,

$$q - 1 = \frac{\text{Var}(N)}{\langle N \rangle^2} - \frac{1}{\langle N \rangle}. \quad (20)$$

For BD one has  $\text{Var}(N) / \langle N \rangle < 1$ , therefore for it  $q < 1$ . For PD  $\text{Var}(N) / \langle N \rangle = 1$ , i.e.,  $q = 1$  as well. For NBD, where  $\text{Var}(N) / \langle N \rangle > 1$ , one has  $q > 1$ .

In the case of  $q > 1$ , i.e., for NDB, fluctuations of multiplicity  $N$  can be translated into fluctuations of temperature  $T$ . This is possible because, as shown in [15, 32], NBD, which can be written also in the following form,

$$P(N) = \frac{\Gamma(N+k)}{[\Gamma(N+1)\Gamma(k)]} \cdot \gamma^k (1+\gamma)^{-k-N},$$

can be obtained from the Poisson multiplicity distribution,  $P(N) = \frac{\bar{N}^N}{N!} e^{-\bar{N}}$ , by fluctuating mean multiplicity  $\bar{N}$  using gamma distribution<sup>3</sup>,

$$f(\bar{N}) = \frac{\gamma^k \bar{N}^{k-1}}{\Gamma(k)} \cdot e^{-\gamma \bar{N}} \quad \text{with} \quad \gamma = \frac{k}{\langle \bar{N} \rangle}.$$

Identifying fluctuations of mean multiplicity  $\bar{N}$  with fluctuations of temperature  $T$ , one can express the above observation via fluctuations of temperature. Namely, noticing that

$$\bar{\beta} = \frac{\bar{N}}{U}, \quad \langle \bar{N} \rangle = U \langle \bar{\beta} \rangle \quad \text{and} \quad \gamma = \frac{k}{U \langle \bar{\beta} \rangle},$$

one can rewrite the gamma distribution for mean multiplicity,  $f(\bar{N})$ , as a gamma distribution of mean inverse temperature  $\bar{\beta}$ ,

$$f(\bar{\beta}) = \frac{k}{\langle \bar{\beta} \rangle \Gamma(k)} \left( \frac{k \bar{\beta}}{\langle \bar{\beta} \rangle} \right)^{k-1} \exp\left(-\frac{k \bar{\beta}}{\langle \bar{\beta} \rangle}\right) = \frac{\left( \frac{1}{q-1} \frac{\bar{\beta}}{\langle \bar{\beta} \rangle} \right)^{\frac{1}{q-1}-1}}{(q-1) \langle \bar{\beta} \rangle \Gamma\left(\frac{1}{q-1}\right)} \exp\left(-\frac{1}{q-1} \frac{\bar{\beta}}{\langle \bar{\beta} \rangle}\right). \quad (21)$$

<sup>3</sup>We have two types of average here:  $\bar{X}$  means average value in a given event whereas  $\langle X \rangle$  denotes averages over all events (or ensembles).

And this is precisely the gamma distribution describing temperature fluctuations, derived, used and investigated in *superstatistics* [29, 30, 28]. When convoluted with the Boltzmann-Gibbs distribution, with  $\beta$  as scale parameter, it carries it into Tsallis distribution with parameter

$$q = 1 + \frac{\text{Var}(\tilde{\beta})}{\langle \tilde{\beta} \rangle} \quad (22)$$

replacing previous Eq. (20) and now denoting the strength of temperature fluctuations.

### 2.5. Preferential attachment

So far, to get  $q > 1$  we were demanding the existence in the system some form of intrinsic (i.e., nonstatistical) fluctuations. However, the same effect can be obtained if the system exhibits correlations of the preferential attachment type, corresponding to the "rich-get-richer" phenomenon in networks [33, 34, 35], and if the scale parameter depends on the variable under consideration. If

$$T \rightarrow T'_0(E) = T_0 + (q - 1)E, \quad (23)$$

then the probability distribution function,  $f(E)$ , is given by an equation the solution of which is a Tsallis distribution (again, with  $q > 1$ ):

$$\frac{df(E)}{dE} = -\frac{1}{T'_0(E)}f(E) \implies f(E) = \frac{2-q}{T_0} \left[ 1 - (1-q)\frac{E}{T_0} \right]^{\frac{1}{1-q}}. \quad (24)$$

For  $T'_0(E) = T_0$  one gets again the usual exponential distribution. This approach was also applied to an analysis of multiparticle production processes in [35].

The "preferential attachment" can be also obtained from superstatistics. As shown above, fluctuations of multiplicity  $N$  are equivalent to the results of application of superstatistics, where the convolution

$$f(E) = \int g(T) \exp\left(-\frac{E}{T}\right) dT \quad (25)$$

becomes a Tsallis distribution, Eq. (2), for

$$g(T) = \frac{1}{\Gamma(n)T} \left(\frac{nT_0}{T}\right)^n \exp\left(-\frac{nT_0}{T}\right). \quad (26)$$

Differentiating Eq. (25) one obtains

$$\frac{df(E)}{dE} = -\frac{1}{T(E)}f(E) \quad \text{where} \quad T(E) = T_0 + \frac{E}{n}. \quad (27)$$

This is nothing else but the "preferential attachment" case, again resulting in a Tsallis distribution, which for  $T(E) = T_0$  becomes BG distribution, cf. Eq. (24)<sup>4</sup>.

### 2.6. Multiplicative noise

Tsallis distribution can also be obtained from *multiplicative noise* [29, 30] defined by the following Langevin equation [30],

$$\frac{dp}{dt} + \gamma(t)p = \xi(t). \quad (28)$$

Here  $\gamma(t)$  and  $\xi(t)$  denote stochastic processes corresponding to, respectively, multiplicative and additive noises. The resulting Fokker-Planck equation for distribution function  $f$ ,

$$\frac{\partial f}{\partial t} = -\frac{\partial (K_1 f)}{\partial p} + \frac{\partial^2 (K_2 f)}{\partial p^2}, \quad (29)$$

<sup>4</sup>This is not the only place where such a form of  $T(E)$  appears. For example, in [22] it occurs in a description of the thermalization of quarks in a quark-gluon plasma by a collision process treated within Fokker-Planck dynamics.

where

$$K_1 = \langle \xi \rangle - \langle \gamma \rangle p \quad \text{and} \quad K_2 = \text{Var}(\xi) - 2\text{Cov}(\xi, \gamma)p + \text{Var}(\gamma)p^2, \quad (30)$$

has a stationary solution  $f$ , which satisfies

$$\frac{d(K_2 f)}{dp} = K_1 f. \quad (31)$$

If there is no correlation between noises and no drift term due to the additive noise, i.e., for  $\text{Cov}(\xi, \gamma) = \langle \xi \rangle = 0$  [36], the solution of this equation is a Tsallis distribution for  $p^2$ ,

$$f(p) = \left[ 1 + (q-1) \frac{p^2}{T} \right]^{\frac{1}{1-q}} \quad \text{with} \quad T = \frac{2\text{Var}(\xi)}{\langle \gamma \rangle}; \quad q = 1 + \frac{2\text{Var}(\gamma)}{\langle \gamma \rangle}. \quad (32)$$

If we insist on a solution in the form of Eq. (1),

$$f(p) = \left[ 1 + \frac{p}{nT} \right]^n \quad \text{with} \quad n = \frac{1}{q-1}, \quad (33)$$

then the condition to be satisfied has the form:

$$K_2(p) = \frac{nT + p}{n} \left[ K_1(p) - \frac{dK_1(p)}{dp} \right]. \quad (34)$$

One then gets a Tsallis distribution (33) but now

$$n = 2 + \frac{\langle \gamma \rangle}{\text{Var}(\gamma)} \quad \text{or} \quad q = 1 + \frac{\text{Var}(\gamma)}{\langle \gamma \rangle + 2\text{Var}(\gamma)} \quad (35)$$

and  $T$  becomes a  $q$ -dependent quantity (reminiscent of effective temperature  $T_{eff}$  as introduced by us in [15]):

$$T(q) = (2-q)[T_0 + (q-1)T_1] \quad \text{with} \quad T_0 = \frac{\text{Cov}(\xi, \gamma)}{\langle \gamma \rangle}, \quad T_1 = \frac{\langle \xi \rangle}{2\langle \gamma \rangle}. \quad (36)$$

## 2.7. From Shannon entropy to Tsallis distribution

As shown in [37], a Tsallis distribution emerges in a natural way from the usual Shannon entropy,  $S$  (for some probability density  $f(x)$ ), by means of the usual MaxEnt approach, if only one imposes the right constraint provided by some function of  $x$ ,  $h(x)$ :

$$S = - \int dx f(x) \ln[f(x)] \quad \text{with constraint} \quad \langle h(x) \rangle = \int dx f(x) h(x) = \text{const}. \quad (37)$$

This approach contains the same information as that based on Tsallis entropy. In fact, one can either use Tsallis entropy with relatively simple constraints, or the Shannon entropy with rather complicated ones (cf., for example, a list of possible distributions one can get in this way [38]). One gets:

$$f(x) = \exp[\lambda_0 + \lambda h(x)], \quad (38)$$

with constants  $\lambda_0$  and  $\lambda$  calculated from the normalization of  $f(x)$  and from the constraint equation. A constraint

$$\langle z \rangle = z_0 = \frac{q-1}{2-q} \quad \text{where} \quad z = \ln \left[ 1 - (1-q) \frac{E}{T_0} \right], \quad (39)$$

results in a Tsallis distribution (remember that  $f(z)dz = f(E)dE$ ),

$$f(z) = \frac{1}{z_0} \exp\left(-\frac{z}{z_0}\right) \implies f(E) = \frac{1}{(1+z_0)T_0} \left(1 + \frac{z_0}{1+z_0} \frac{E}{T_0}\right)^{-\frac{1+z_0}{z_0}} = \frac{2-q}{T_0} \left[1 - (1-q) \frac{E}{T_0}\right]^{\frac{1}{1-q}}. \quad (40)$$

To obtain  $T_0$ , one has to assume the knowledge of  $\langle E \rangle$  (this would be the an only constraint in the case of BG distribution but here it is additional condition to be accounted for).

Although at the moment there is no clear understanding of the physical meaning of the constraint (39) (except its obvious usefulness in getting Eq. (40)), it seems to be a natural one from the point of view of the multiplicative noise approach represented by Eq. (28). That is because there is a connection between the kind of noise occurring in Eq. (28) and the condition imposed in the MaxEnt approach [39]. Namely, for processes described by an additive noise,  $dx/dt = \xi(t)$ , the natural condition is that imposed on the arithmetic mean,  $\langle x \rangle = c + \langle \xi \rangle t$ , and it results in exponential distributions. For the multiplicative noise,  $dx/dt = x\gamma(t)$ , the natural condition is that imposed on the geometric mean,  $\langle \ln x \rangle = c + \langle \gamma \rangle t$ , which results in a power law distribution [40]. It seems, therefore, that condition (39) combines both possibilities and leads to a quasi-power law Tsallis distribution combining both types of behavior.

### 2.8. Tsallis and QCD

Recent high energy experiments from the Large Hadron Collider at CERN (CMS [1], ATLAS [2] and ALICE [3]) provided distributions of transverse momenta measured in the previously unprecedented range  $p_T \leq 180$  GeV. The measured cross section then spans the range of  $\sim 14$  orders of magnitude. In [41] it was shown that all these data can be successfully fitted by Tsallis distribution (1) or (2), which is since then widely used in this case (cf., Fig. 2 as example). This caused question, how is it possible because this is the usual domain reserved for the purely perturbative QCD approach? In [42] it was demonstrated that:

- Starting from the pure QCD partonic picture of elementary collisions ("hard" scatterings between quarks and gluons of incoming protons proceeding with high momentum transfer) one gets power index  $n \simeq 4 - 4.5$ . However one observes hadrons which are formed from quarks and gluons by means of complicated branching and fragmentation processes. It turns out that all these processes can be parameterized in a relatively simple way and one can easily reproduce  $n \simeq 7 - 8$  as observed in the experiment (depending on the energy of collision).
- However, in this way one reproduces properly only the power index  $n$  (or  $q$ ) and resulting distribution is of pure power-like type,  $\sim 1/p_T$ , diverging for  $p_T \rightarrow 0$  instead of being exponential there. To get a Tsallis distribution one has to make the same phenomenological step as that proposed to obtain Eq. (1): to replace  $p_T \rightarrow p_{T0} + p_T$  in the denominator of  $1/p_T$ . So far, the only rationale behind this is that, in the QCD approach, large  $p_T$  partons probe small distances (with small cross sections). With diminishing of  $p_T$ , these distances become larger (and cross sections are increasing) and, eventually, they start to be of the order of the nucleon size (actually it happens around  $p_T \sim p_{T0}$ ). At this moment the cross section should stop rising, i.e., it should not depend anymore on the further decreasing of transverse momentum  $p_T$ . This can be modelled by introducing the constant term as above. Effectively one then has  $(1/p_T)^{-n} \rightarrow [p_{T0} \cdot (1 + p_T/p_{T0})]^{-n}$ . The scale parameter  $p_{T0}$  can then be identified with that in Fig. 1. In a Tsallis fit we use one formula for the whole phase space with  $p_{T0}$  becoming  $T$  in the exponent for small transverse momenta, and scale parameter for large  $p_T$ . Usually one uses an exponential formula for  $p_T < p_{T0}$  and power for  $p_T > p_{T0}$  and  $p_{T0}$  separates the two parts of phase space.

### 3. Log-periodic oscillations

So far, we have presented possible derivations and applications of relatively simple form of Tsallis distribution with Fig. 2 as an example of its apparent success in fitting even the most demanding data so far. However, closer inspection of Fig. 2 shows that ratio of data/fit, usually used to estimate the quality of fits, is not flat but shows some kind of clearly visible oscillations of *log-periodic* character, cf. Fig. 3. The first conjunction in such a case is that parameters used were not chosen in an optimal way. However, it turns out that these oscillations cannot be eliminated by any suitable changes of parameters ( $q, T$ ) or ( $m, T$ ) in Eqs. (2) or (1), respectively. Here we shall concentrate only on data from the CMS experiment [1], data from ATLAS [2] lead to identical conclusions. One also has to realize that to really see these oscillations one needs a rather large domain in  $p_T$ . Therefore, albeit similar effects can also be seen at lower energies, they are not so pronounced as here and will not be discussed here. Assuming that this is not an experimental artifact one has to admit that it tells us that the Tsallis formula used is too simple. There still remains something hidden in data which has, so far, avoided to be disclosed, and which can signal some genuine dynamical effect which is worth been investigating in more detail.



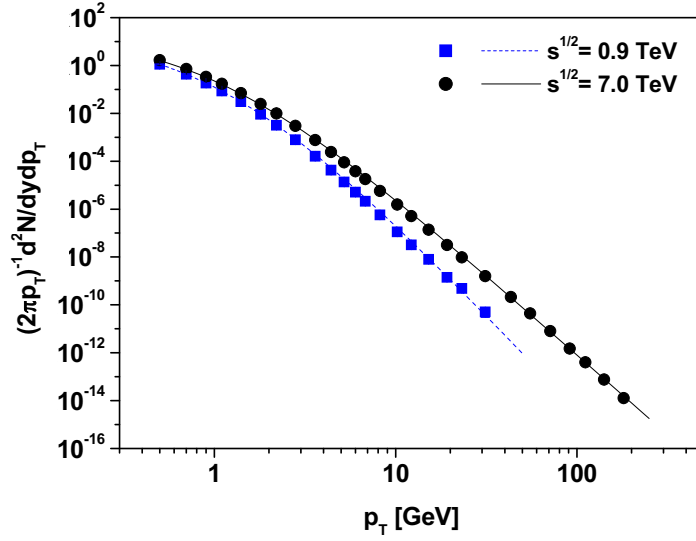


Figure 2. (Color online) Fit to large  $p_T$  data for  $pp$  collisions at 0.9 and 7 TeV from CMS experiment using distribution (1) [1]. Parameters used are, respectively, ( $T = 0.135$ ,  $m = 8$ ) and ( $T = 0.145$ ,  $m = 6.7$ ).

Rather than look for another distribution we shall keep to Tsallis distribution and attempt to improve it accordingly to account for effects of these *log-periodic oscillations* observed in data. Because we have two parameters here, power index  $n$  and scale (“temperature”)  $T$ , the natural approach is to modify one of them. Because data, which we shall analyze below, are presented at midrapidity (i.e., for  $y \sim 0$  and longitudinal momentum  $p_L \sim 0$ ) and for large transverse momenta,  $p_T \gg \mu$ , the energy  $E$  of a produced particle is essentially equal to its transverse momentum  $p_T$ , which we shall use in what follows.

### 3.1. Quasi-power laws with complex power indices

First notice that log-periodic oscillations are ubiquitous in systems described in general by power distributions [44]. Usually they suggest existence of some scale-invariant hierarchical fine-structure in the system and indicate its possible multifractality [45]. In the context of nonextensive statistical mechanics log-periodic oscillations have been first observed while analyzing the convergence to the critical attractor of dissipative maps [46] and restricted random walks [47]. In the case of pure power like distributions the only parameter to manipulated was the power index. It was then natural to modify this parameter by allowing it to be complex [44, 45].

For quasi-power Tsallis distributions this idea was first investigated in detail by us in [48]. Here we shall apply it to multiparticle data as mentioned before. The complex power index results in effective *dressing* the original distribution by multiplying it by a log-oscillating function, usually taken in the form of:

$$R(E) = a + b \cos [c \ln(E + d) + f]. \quad (41)$$

In [48] we derived such a factor for a Tsallis distribution. For completeness, we recapitulate the main points of this derivation. In general, if one deals with a scale invariant function,  $O(x)$ , i.e., if

$$O(\lambda x) = \mu O(x), \quad (42)$$

then it must have power law behavior,

$$O(x) = Cx^{-m} \quad \text{with} \quad m = -\frac{\ln \mu}{\ln \lambda}. \quad (43)$$

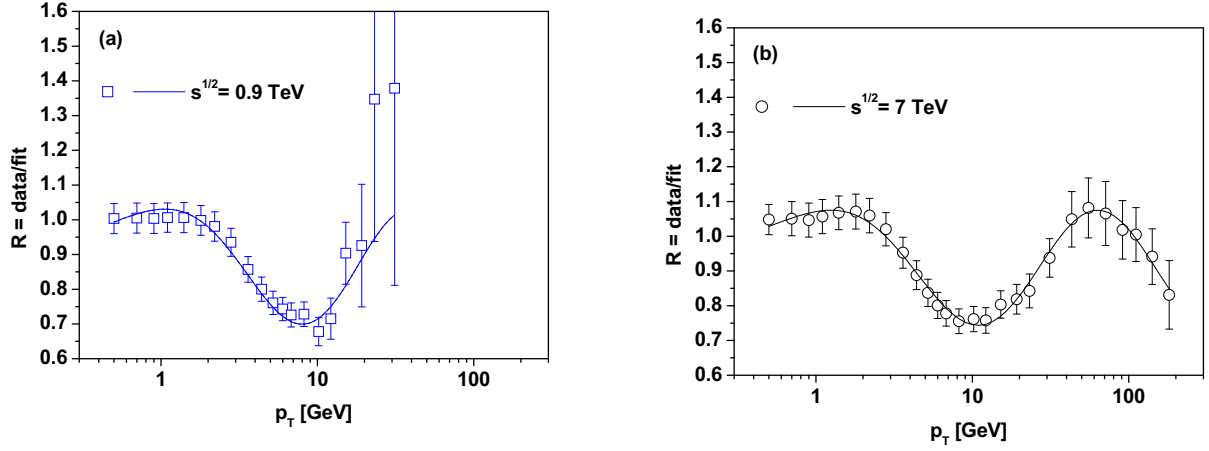


Figure 3. (Color online) Fit to  $p_T$  dependence of data/fit ratio for results from Fig. 2. Parameters of function  $R$  used (41) here are, respectively:  $a = 0.865$ ,  $c = 2.1$  for 0.9 TeV and  $a = 0.909$ ,  $c = 1.86$  for 7 TeV, whereas for both energies  $b = 0.166$ ,  $d = 0.948$  and  $f = -1.462$ .

It means that, in general, one can write that ( $k$  is an arbitrary integer)

$$\mu\lambda^m = 1 = e^{i2\pi k} \implies m = -\frac{\ln \mu}{\ln \lambda} + i\frac{2\pi k}{\ln \lambda}. \quad (44)$$

One must now find whether the Tsallis distribution has a similar property and under what conditions. To this end we start from differential  $df(E)/dE$  of a Tsallis distribution  $f(E)$  with power index  $n$ , cf. Eq. (27), and write it for finite differences [48],

$$\delta E = \alpha(nT + E), \quad (45)$$

where  $\alpha n < 1$  is a new parameter. This leads to the following scale invariant relation,

$$g[(1 + \alpha)x] = (1 - \alpha n)g(x) \quad (46)$$

where

$$x = 1 + \frac{E}{nT}. \quad (47)$$

This means then that, in general, one can write Eq. (1) in the form:

$$g(x) = x^{-m_k}, \quad m_k = -\frac{\ln(1 - \alpha n)}{\ln(1 + \alpha)} + ik\frac{2\pi}{\ln(1 + \alpha)}. \quad (48)$$

The power index in Eq. (48) (and therefore also in Eq. (1)) becomes a complex number, its imaginary part is signaling a hierarchy of scales leading to the log-periodic oscillations.

If we limit ourselves to  $k = 0$ , one recovers the usual real power law solution and  $m_0$  corresponds to fully continuous scale invariance. However, in this case the power law exponent  $m_0$  still depends on  $\alpha$  and increases with it roughly as

$$m_0 \simeq n + \frac{n}{2}(n+1)\alpha + \frac{n}{12}(4n^2 + 3n - 1)\alpha^2 + \frac{n}{24}(6n^3 + 4n^2 - n + 1)\alpha^3 + \dots \quad (49)$$

The usual Tsallis distribution is recovered only in the limit  $\alpha \rightarrow 0$ .

In general one has

$$g(x) = \sum_{k=0} w_k \cdot \text{Re}(x^{-m_k}) = x^{-\text{Re}(m_k)} \sum_{k=0} w_k \cdot \cos[\text{Im}(m_k) \ln(x)]. \quad (50)$$

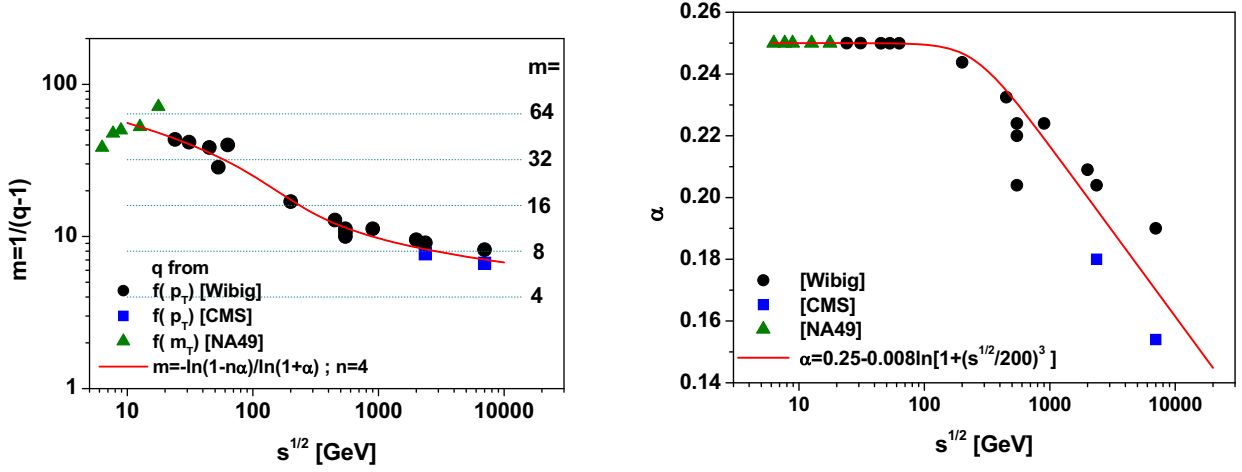


Figure 4. (Color online) Left panel: The energy dependence of  $m = m_0$  deduced from CMS [1] and NA49 [43] data and from a compilation [Wibig] [17]. Right panel: The energy dependence of parameter  $\alpha$  present in  $m_0$  plotted in left panel.

This is a general form of a Tsallis distribution for complex values of the nonextensivity parameter  $q$ . It consists of the usual Tsallis form (albeit with a modified power exponent) and a dressing factor which has the form of a sum of log-oscillating components, numbered by parameter  $k$ . Because we do not know *a priori* the details of dynamics of processes under consideration (i.e., we do not know the weights  $w_k$ ), in what follows we only use  $k = 0$  and  $k = 1$  terms. We obtain approximately,

$$g(E) \simeq \left(1 + \frac{E}{nT}\right)^{-m_0} \cdot \left\{ w_0 + w_1 \cos \left[ \frac{2\pi}{\ln(1+\alpha)} \ln \left( 1 + \frac{E}{nT} \right) \right] \right\}. \quad (51)$$

In this case one could expect that parameters in general modulating factor  $R$  in Eq. (41) could be identified as follows:

$$a = w_0, \quad b = w_1, \quad c = \frac{2\pi}{\ln(1+\alpha)}, \quad d = nT, \quad f = -c \cdot \ln(nT). \quad (52)$$

Comparison of the fit parameters of the oscillating term  $R$  in Eq. (41) with Eq. (48) clearly shows that the observed frequency, here given by the parameter  $c$ , is more than an order of magnitude smaller than the expected value equal to  $2\pi / \ln(1+\alpha)$  for any reasonable value of  $\alpha$ . To explain this, notice that in our formalism leading to Eq. (51) only one evolution step is assumed, whereas in reality we have a whole hierarchy of  $\kappa$  evolutions. This results (cf. [48]) in the scale parameter  $c$  being  $\kappa$  times smaller than in (51),

$$c = \frac{2\pi}{\kappa \ln(1+\alpha)}. \quad (53)$$

Experimental data indicate that  $\kappa \simeq 22$  (for  $\alpha \simeq 0.15$  and  $c \simeq 2$ ).

From Eq.(48) we see that  $m_0 > n$ . This suggests the following explanation of the difference seen between prediction from theory and the experimental data: the measurements in which log-periodic oscillations appear underestimate the true value that follows from the underlying dynamics which leads to the smooth Tsallis distribution. As an example consider the  $m_0$  dependence on  $\alpha$ , assuming the initial slope  $n = 4$  (this is the value of  $n$  expected from the pure QCD considerations for partonic interactions [42]). The energy behavior of the power index  $m_0$  in the Tsallis part is shown in the left panel of Fig. 4, whereas the energy dependence of the parameter  $\alpha$  contained in  $m_0$  is shown in the right panel of Fig. 4.

### 3.2. Quasi-power laws with log-periodic scale parameter $T$

The phenomenon of log-periodic oscillations observed in data can also be explained in a different way. We can keep the nonextensivity parameter  $q$  real (as in the original Tsallis distribution) but allow the scale parameter  $T$  to

oscillate in a specific way, as displayed in Fig. 5. These oscillations can be fitted by a formula similar to Eq. (41), with generally energy dependent fit parameters  $(\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{f})$ :

$$T = \bar{a} + \bar{b} \sin \left[ \bar{c} \left( \ln(E + \bar{d}) + \bar{f} \right) \right]. \quad (54)$$

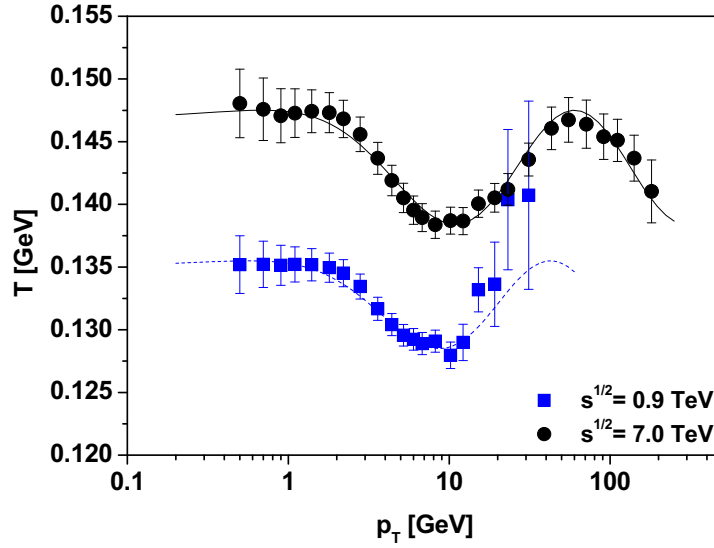


Figure 5. (Color online) The  $T = T(p_T)$  for Eq. (2) for which  $R = 1$ . Parameters used are:  $\bar{a} = 0.132$ ,  $\bar{b} = 0.0035$ ,  $\bar{c} = 2.2$ ,  $\bar{d} = 2.0$ ,  $\bar{f} = -0.5$  for 0.9 TeV and  $\bar{a} = 0.143$ ,  $\bar{b} = 0.0045$ ,  $\bar{c} = 2.0$ ,  $\bar{d} = 2.0$ ,  $\bar{f} = -0.4$  for 7 TeV.

To get such behavior we start from the well known stochastic equation for the temperature evolution [49], which in the Langevin formulation (allowing for an energy dependent noise,  $\xi(t, E)$ ) has the form:

$$\frac{dT}{dt} + \frac{1}{\tau} T + \xi(t, E) T = \Phi, \quad (55)$$

$\tau$  is relaxation time which, for a while, we keep constant. For the time dependent  $E = E(t)$  it reads:

$$\frac{dT}{dE} \frac{dE}{dt} + \frac{1}{\tau} T + \xi(t, E) T = \Phi. \quad (56)$$

In the scenario of *preferential attachment* known from the growth of networks, cf. Section 2.5, evolution equations as given by Eqs. (24) and (27) can be derived from master equation  $df(E, t)/dt = -f(E, t)$  for the growth of network  $dE/dt$  given by [33]

$$\frac{dE}{dt} = \frac{E}{n} + T \quad (57)$$

( $n$  coincides with power index in Eq. (1)). This will therefore be the equation we shall use in what follows<sup>5</sup>. With it one can write Eq. (56) as

$$\left( \frac{E}{n} + T \right) \frac{dT}{dE} + \frac{1}{\tau} T + \xi(t, E) T = \Phi. \quad (58)$$

<sup>5</sup>Notice that in the usually multiplicative noise scenario described by  $\gamma(t)$ , not discussed here, one has  $\frac{dE}{dt} = \gamma(t)E + \xi(t)$ .

This can be subsequently transformed to

$$\left(\frac{1}{n} + T e^{-\ln E}\right) \frac{dT}{d(\ln E)} + \frac{1}{\tau} T + \xi(t, E) T = \Phi \quad (59)$$

and, after differentiating, to

$$\left(\frac{1}{n} + T e^{-\ln E}\right) \frac{d^2 T}{d(\ln E)^2} + \left[\frac{dT}{d(\ln E)}\right]^2 e^{-\ln E} - \left[T e^{-\ln E} - \frac{1}{\tau} - \xi(t, E)\right] \frac{dT}{d(\ln E)} + T \frac{d\xi(t, E)}{d(\ln E)} = 0. \quad (60)$$

For large  $E$  (i.e., neglecting terms  $\propto 1/E$ ) one obtains the following equation for  $T$ :

$$\frac{1}{n} \frac{d^2 T}{d(\ln E)^2} + \left[\frac{1}{\tau} + \xi(t, E)\right] \frac{dT}{d(\ln E)} + T \frac{d\xi(t, E)}{d \ln E} = 0. \quad (61)$$

To proceed further one has to specify the energy dependence of the noise  $\xi(t, E)$ . We assume that it increases logarithmically with energy in the following way,

$$\xi(t, E) = \xi_0(t) + \frac{\omega^2}{n} \ln E \quad (62)$$

(where  $n$  is, again, power index from Eq. (1) and  $\omega$  is a new parameter). For this choice of noise Eq. (61) is just an equation for the damped hadronic oscillator and has a solution in the form of a log-periodic oscillation of temperature with frequency  $\omega$ :

$$T = C \exp \left\{ -n \cdot \left[ \frac{1}{2\tau} + \frac{\xi(t, E)}{2} \right] \ln E \right\} \cdot \sin(\omega \ln E + \phi). \quad (63)$$

The phase shift parameter  $\phi$  depends on the unknown initial conditions and is therefore an additional fitting parameter. Averaging the noise fluctuations over time  $t$  and taking into account that the noise term cannot on average change the temperature (cf. Eq. (55) in which  $\langle dT/dt \rangle = 0$  for  $\Phi = 0$ ), i.e., that

$$\frac{1}{\tau} + \langle \xi(t, E) \rangle = 0, \quad (64)$$

we have

$$T = \bar{a} + \frac{b'}{n} \sin(\omega \ln E + \phi). \quad (65)$$

The amplitude of oscillations,  $b'/n$ , comes from the assumed behavior of the noise as given in Eq. (62). Notice that for large  $n$ , the energy dependence of the noise disappears. It means that, because, in general,  $n$  decreases with energy [42], one can expect only negligible oscillations for lower energies but increase with the energy.

This should now be compared with the parametrization of  $T(p_T)$  given by Eq. (54) and used to fit data in Fig.5. Looking at parameters we can see that only a small amount of  $T$  (of the order of  $\bar{b}/\bar{a} \sim 3\%$ ) comes from the stochastic process with energy dependent noise, whereas the main contribution emerges from the usual energy-independent Gaussian white noise.

The above oscillating  $T$  needed to fit the log-periodic oscillations seen in data can be obtained in yet another way. So far we were assuming that the noise  $\xi(t, E)$  has the form of Eq. (62) and, at the same time, we were keeping the relaxation time  $\tau$  constant. In fact, we could equivalently assume the energy  $E$  independent white noise,  $\xi(t, E) = \xi_0(t)$ , but allow for the energy dependent relaxation time. Assuming it in the form,

$$\tau = \tau(E) = \frac{n\tau_0}{n + \omega^2 \ln E}, \quad (66)$$

results in the following time evolution of the temperature,

$$T(t) = \langle T \rangle + [T(t=0) - \langle T \rangle] E^{-t\omega^2/n} \exp\left(-\frac{t}{\tau_0}\right). \quad (67)$$

It gradually approaches its equilibrium value,  $\langle T \rangle$ , and reaches it sooner for higher energies.

### 3.2.1. Log-periodic oscillations: summary

To summarize, we have presented two possible mechanisms which could result in the log-periodic oscillations apparently present in data for transverse momentum distributions observed in LHC experiments. In both cases one uses a Tsallis formula (either in the form of Eq. (1) or Eq. (2)), with main parameters  $m$  - the scaling exponent (or nonextensivity  $q = 1 + 1/m$ ) and  $T$  - the scale parameter (temperature).

- In the first approach, our Tsallis distribution is decorated by an oscillating factor. This is done by changing in Eq. (27) differentials by finite differences,  $dE \rightarrow \delta E = \alpha(nT + E)$ , where the new parameter  $\alpha < 1/n$  regulates the smallness of  $\delta E$ . As results, we get for  $x = 1 + E/(nT)$  the scale invariant relation,  $g[(1 + \alpha)x] = (1 - \alpha n)g(x)$ . This, in turn, means that power index  $m$  (and also nonextensivity parameter  $q$ ) becomes a complex number, of which the imaginary part describes a hierarchy of scales leading to the log-periodic oscillations. The scale parameter  $T$  remains unaltered.

It should be mentioned at this point that complex  $q$  inevitably also means complex heat capacity  $C = 1/(1 - q)$  (c.f., [50, 16, 51]). Such complex (frequency dependent) heat capacities (meaning relaxing temperature) are widely known and investigated, see [52].

- In the second approach, it is the other way around, i.e., whereas  $m = n$  remains untouched, the scale parameter  $T$  is now oscillating. From Eq. (65) one can see that  $T = T(n = 1/(1 - q), E)$  and as a function of nonextensivity parameter  $q$  it continues our previous efforts to introduce an effective temperature into the Tsallis distribution,  $T_{eff} = T(q)$  [15, 53, 39] (but here in a much more general form). The two possible mechanisms resulting in such  $T$  were outlined: (i) - the energy dependent noise connected with the constant relaxation time, or else, (ii) - the energy independent white noise, but with energy dependent relaxation time.

At the present level of investigation, we are not able to indicate which of the two possible mechanisms presented here (complex  $q$  or oscillating  $T$ ) and resulting in log-periodic oscillations is preferred. This would demand more detailed studies on the possible connections with dynamical pictures. For example, as discussed long time ago by studying apparently similar effects in some exclusive reactions using the QCD Coulomb phase shift idea [54]. The occurrence of some kind of complex power exponents was noticed there as well, albeit on completely different grounds than in our case. A possible link with our present analysis would be very interesting but would demand an involved and thorough analysis.

## 4. Summary and conclusions

We present examples of a possible mechanisms resulting in the quasi-power distributions exemplified by Tsallis distribution Eqs. (1) and (2). Our presentation is limited to approaches not derived from nonextensive thermodynamic connections of this distribution [19, 20, 23].

It was shown that statistical physics consideration, as well as "induced partition process", results in a Tsallis distribution with  $q = (N - 3)/(N - 2) < 1$ , Eq. (3). To get  $q > 1$  one has to allow for fluctuations of the multiplicity  $N$ . They modify the parameter  $q$  which is now  $q = 1 + \text{Var}(N)/\langle N \rangle^2 - 1/\langle N \rangle$ , cf. Eq. (20). The conditional probability for BG distribution results again in Eq. (3).

We proposed and discussed two possible mechanisms which would allow quasi-power law distributions to which Tsallis distribution belongs, to describe data showing a log-periodic "decoration" of simple power law distributions. One is a generalization of Tsallis distribution to real power  $n$  (it can be regarded as generalization of such well known distributions as Snadecor distribution (with  $n = (\nu + 2)/2$  with integer  $\nu$ , for  $\nu \rightarrow \infty$  it becomes an exponential distribution), can be extended to complex nonextensivity parameter). The other is introducing a specific, log-periodic oscillating scale parameter, effective temperature  $T_{eff}$ , generalizing our previous results in this field presented and used in [15, 39, 53].

Different derivations of Tsallis distributions turn out to be, in a sense, equivalent. Fluctuations of multiplicity  $N$  are equivalent to results fluctuations of  $T$  (which is the basis of of superstatistics); on the other hand, from a superstatistics formula one can get the preferential attachment, cf. Sections 2.4 and 2.5.

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